Life-time of minimal tubes and coefficients of univalent functions in a circular ring

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Abstract. We obtain various estimates of the life-time of two-dimensional minimal tubes in $\mathbb{R}^3$ by potential theory methods.

1. Introduction.

Let $x = (x_1, x_2, \ldots, x_n, x_{n+1})$ be a point in Euclidean space $\mathbb{R}^{n+1}$ with the *time* axis Ox_{n+1} and M be a p-dimensional Riemannian manifold, $2 \leq p \leq n$.

Definition 1. We say that a surface $\mathcal{M} = (M, \mathbf{u})$ given by C^2-immersion $\mathbf{u} : M \to \mathbb{R}^{n+1}$ is a *tube* with the projection interval $\tau(\mathcal{M}) \subset Ox_{n+1}$, if (i) for any $\tau \in \tau(\mathcal{M})$ the sections $\Sigma_\tau = f(\mathcal{M}) \cap \Pi_\tau$ by hyperplanes $\Pi_\tau = \{x \in \mathbb{R}_1^{n+1} : x_{n+1} = \tau\}$ are not empty compact sets; (ii) for $\tau', \tau'' \in \tau(\mathcal{M})$ any part of $\mathcal{M}$ situated between two different $\Pi_{\tau'}$ and $\Pi_{\tau''}$ is a compact set.

Definition 2. A surface $\mathcal{M}$ is called *minimal* if the mean curvature of $\mathcal{M}$ vanishes everywhere.

It is the well known fact (see [5], p.331) that the minimality condition of $\mathcal{M}$ is equivalent to that all coordination functions of the immersion $\mathbf{u}$ are harmonic. For this reason, the two-dimensional minimal tubes can be considered as direct analog of the closed relative string conception in the modern nuclear physics (cf. [2]). This approach was proposed by V.M.Miklyukov and the author in [7] for an arbitrary dimension p.

From this point of view many intrinsic geometric invariants of $\mathcal{M}$ have the natural physical meaning. Namely, the length of the projection interval $|\tau(\mathcal{M})|$ can be interpreted as a *life-time* of the tube $\mathcal{M}$.

To introduce the following important characteristic we denote by ν the unit normal to Σ_τ with respect to $\mathcal{M}$ which is co-directed with the time-axis Ox_{n+1}. Then by virtue of the harmonicity of the coordinate functions $u_k(m) = x_k \circ \mathbf{u}(m)$,

$1 \leq k \leq n+1$, the flow integrals

$$J_k = \int_{\Sigma_\tau} \langle \nabla u_k, \nu \rangle \, d\Sigma$$

are independent of $\tau \in \tau(\mathcal{M})$. Here $d\Sigma$ is the 1-Hausdorff measure along Σ_τ.

Definition 3. We call $Q(\mathcal{M}) = (J_1, J_2, \ldots, J_{n+1}) \in \mathbb{R}^{n+1}$ the *full flow-vector* of $\mathcal{M}$.

We notice the positiveness of J_{n+1} as a consequence of the choice of ν direction. Moreover, $Q(\mathcal{M})$ is an 1-homogeneous functional of $\mathcal{M}$ under the homotheties group action in $\mathbb{R}^{n+1}$. Let us denote by $\alpha(\mathcal{M})$ the angle between $Q(\mathcal{M})$ and the time-axis Ox_{n+1}.

In this paper we are interested in the following question: What sufficient conditions yield the finiteness of the time-life of a two-dimensional minimal tube? As it shown in the series of papers [6]–[8], in the case $p \geq 3$ this quantity is always finite and the following estimation holds

$$|\tau(\mathcal{M})| \leq \varrho(\mathcal{M})c_p,$$

where c_p depends only on p, and $\varrho(\mathcal{M})$ is the smallest diameter of sections Σ_τ. The last relationship is sharp and the equality occurs if and only if $\mathcal{M}$ is a minimal surface of revolution.

A special feature of the two-dimensional case is that there exist tubes with finite as well as infinite values of the life-time. Indeed, a family of slanted minimal surfaces with circular cross-sections Σ_τ was discovered by B. Riemann [10]. Some other recent examples can also be found in [4].

In this paper we prove

Theorem 1. *Let $\mathcal{M}$, $\dim \mathcal{M} = 2$ be a minimal two-connected tube with univalent Gaussian mapping. If the angle $\alpha(\mathcal{M})$ is different from zero, then the life-time $|\tau(\mathcal{M})|$ of $\mathcal{M}$ is finite and*

$$\tau(\mathcal{M}) \leq \frac{\pi \|Q\| \cos \alpha(\mathcal{M})}{\ln \tan(\frac{\pi}{4} + \frac{\alpha}{2})}.$$

Let us denote by $a_0[f]$ the central coefficient of the Laurent decomposition of an holomorphic function $f(z)$ in an annulus $K_R = \{z : 1/R < |z| < R\}$, i.e.

$$a_0[f] \equiv \int_{C_1} \frac{g(\zeta) \, d\zeta}{\zeta}.$$

where C_1 is the unite circle $\{z \in \mathbb{C} : |z| = 1\}$. The following auxiliary assertion is a key ingredient in the proof of Theorem 1.

Theorem 2. *Let $g(z)$ be a univalent holomorphic function defined in the annulus K_R omitting zero. Assume that*

(1) $a_0[g] = \lambda, \qquad a_0[1/g] = -\lambda,$

for some real positive λ. Then

$$(2) \qquad \ln R \leq \ln R_0(\lambda) = \frac{\pi^2}{\ln(\lambda + \sqrt{1 + \lambda^2})}$$

Remark 1. We note that estimate (2) has well asymptotic behaviour for $R \to \infty$ as shows Riemannian example mentioned above. But we cannot now present the sharp value of $R_0(\lambda)$. Nevertheless, it seemed us very probably that the following conjecture is true.

Remark 2. The best upper bound in the left side of (2) is achieved for holomorphic function $g_0(z)$ which provides a conformal map of the annulus K_R onto the plain $\mathbb{C}$ with two slits: $(-1/\alpha; 0)$ and $(\alpha; +\infty)$, for the suitable choice of parameter α.

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2. Proof of Theorem 2

Let $\Gamma = \{C_\rho : 1/R < \rho < R\}$ be a family of all concentric circles $C_\rho = \{z : |z| = \rho\}$ in the annulus K_R. It follows easily from the non-vanishing property of $g(z)$ that the loop C_1 in the integrals (1) may be replaced by an arbitrary circle $C_\rho \in \Gamma$. It follows from the mean value theorem and (1) that for every $\rho \in (1/R; R)$ there exist t_1 and t_2 such that

$$(3) \qquad \operatorname{Re} g(\rho e^{it_1}) = \lambda \quad \text{and} \quad \operatorname{Re} \frac{1}{g(\rho e^{it_2})} = -\lambda.$$

Let $\gamma_\rho = g(C_\rho)$. Then by virtue of the univalence of $g(z)$, the curve γ_ρ is the simple Jordan one. Let $g(\rho e^{it}) = x(t) + iy(t)$ be the representation of γ_ρ. Then we obtain from (3)

$$x(t_1) = \lambda; \qquad x^2(t_2) + y^2(t_2) + \frac{1}{\lambda}x(t_2) = 0.$$

The last relations have the helpful geometric interpretation:

$(\star)$ The curve γ_ρ has a non-empty intersection with the vertical rightline $L_1 = \{z : \operatorname{Re} z = \lambda\}$ and the circle $L_2 = \{z : |z + 1/2\lambda| = 1/2\lambda\}$.

We shall make use the technique from the potential theory (the length-are method). Recall the exact definition. Let E be a family of locally rectifiable curves γ and $\varphi(z) \geq 0$ be a Baire function with the property

$$\int_\gamma \varphi(z)\,|dz| \geq 1,$$

for every $\gamma \in E$. The infimum

$$\operatorname{mod} E = \inf \int \varphi^2(z)\,dx\,dy$$

over all such $\varphi(z)$ is called a *conformal module* of the family E.

Then it is known (see [1]) that mod E is the conformal invariant. As a consequence we obtain in our situation

$$\text{mod } \Gamma = \text{mod } \Gamma_1, \tag{4}$$

where $\Gamma_1 = \{\gamma_\rho : 1/R < \rho < R\}$.

Let us denote by D the two-dimensional domain

$$D = \left\{ z : \text{Re } z < \lambda; \; \left| z + \frac{1}{2\lambda} \right| > \frac{1}{2\lambda} \right\}.$$

Using the $(\star)$-property, we can find for every $\rho \in (1/R; R)$ the continuum $\gamma'_\rho \subset \gamma_\rho$ joining the boundary components of D. Then a family Γ_2 consisting of all continua γ'_ρ is "shorter" than Γ_1 and it follows from Theorem 1.2, [1] that

$$\text{mod } \Gamma_1 \leq \text{mod } \Gamma_2. \tag{5}$$

On the other hand, Γ_2 is the subfamily of $\Gamma(D)$, where the last term means the family of *all* curves joining the boundary components of a domain D. The monotonicity property of infimum and Definition 4 lead to the following inequality

$$\text{mod } \Gamma_2 \leq \text{mod } \Gamma(D). \tag{6}$$

Now, combining the standard fact

$$\text{mod } \Gamma = \frac{\ln R}{\pi} \tag{7}$$

with relations (4), (5) and (6) we arrive at the following inequality

$$\frac{\ln R}{\pi} \leq \text{mod } \Gamma(D).$$

To compute the last module we note that the linear-fractional function

$$f(z) = \frac{1}{\lambda^*} \cdot \frac{z + \lambda^*}{1 - z\lambda^*}$$

maps D onto an annulus $K_1 = \{w : 1 < |w| < 1/\lambda^{*2}\}$, where $\lambda^* = \sqrt{\lambda^2 + 1} - \lambda$. Thus, using the invariance property of conformal module we obtain

$$\frac{\ln R}{\pi} \leq \text{mod}(D) \equiv \frac{2\pi}{\ln\left(1/\lambda^{*2}\right)} = \frac{\pi}{\ln(\lambda + \sqrt{1 + \lambda^2})}.$$

and Theorem 2 is proved.

3. The Gaussian map of two-dimensional minimal tubes and their full-flow vector

In this section we express the full flow-vector of an arbitrary two-dimensional tube $\mathcal{M} \in \mathbb{R}^n$ via Chern-Weierstrass representation for minimal surfaces. Namely, if $\mathcal{M}$ is a two-connected surface then we can arrange that $\mathcal{M}$ is conformally equivalent

to an annulus K_R for the appropriate $R > 1$. Then there exist the corresponding parametrization of $\mathcal{M}$ (see [9]):

$$\mathbf{u}(z) = \mathrm{Re} \int_{z_0}^{z} F(\zeta)\, d\zeta \; : K_R \to \mathbb{R}^n,$$

where

$$F(z) = (\varphi_1(\zeta), \ldots, \varphi_n(\zeta))$$

and $\varphi_i(\zeta)$ are holomorphic functions satisfying the following conditions

$$(8) \qquad\qquad \sum_{i=1}^{n} \varphi_i(\zeta)^2 = 0;$$

and

$$(9) \qquad\qquad \mathrm{Re} \int_{|z|=1} F(\zeta)\, d\zeta = \mathbf{0}.$$

Lemma 1. *Under the above hypotheses we have*

$$(10) \qquad\qquad Q(\mathcal{M}) = \mathrm{Im} \int_{|z|=1} F(\zeta)\, d\zeta.$$

Proof. It sufficient to show that

$$(11) \qquad J_k \equiv \int_{\Sigma_\tau} \langle \nabla u_k, \nu \rangle \, d\Sigma = \mathrm{Im} \int_{|z|=1} \varphi_k(\zeta)\, d\zeta,$$

for every $k = 1, 2, \ldots, n+1$.

To prove (11) we introduce the conjugate to $u_k(z)$ function $v_k(z)$ by

$$v_k^*(z) = \mathrm{Im} \int_{z_0}^{z} \varphi_k(\zeta)\, d\zeta,$$

We notice that $v_k(z)$ in general is a multivalued function. On the other hand, the covariant derivative ∇v_k is well defined and using the properties of Hodge $\star$ - operator we have

$$\int_{\Sigma_\tau} \langle \nabla u_k, \nu \rangle \, d\Sigma = \int_{\Sigma_\tau} \langle \star \nabla u_k, \star \nu \rangle \, d\Sigma = \int_{\Sigma_\tau} \langle \nabla v_k, \star \nu \rangle \, d\Sigma =$$

$$= \int_{\Sigma_\tau} d\, v_k = \mathrm{Im} \int_{|z|=1} \varphi_k(\zeta)\, d\zeta,$$

and (11) is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In our case $n = 2$, Chern-Weierstrass representation can be simplified in the following classic way. Namely, there exist a holomorphic function $f(z)$ and a meromorphic function $g(z)$ which are well defined in the annulus K_R and such that

$$(12) \qquad\qquad F(z) = \left((1 - g^2)f; \; i(1 + g^2)f; \; 2gf \right).$$

Moreover, poles of $g(z)$ coincide with zeros of $f(z)$ and the order of a pole of $g(z)$ is precisely the order of the corresponding zero of $f(z)$. We emphasize that $g(z)$ is a composition of the stereographic projection and Gaussian map of $\mathcal{M}$.

Lemma 2. *In our assumptions*

$$
(13) \qquad\qquad 2fg \equiv \frac{\langle Q(\mathcal{M}), e_3 \rangle}{2\pi z},
$$

and $g(z)$ omits the zero and infinity values.

Proof. We use the method proposed by M. Schiffman in [11]. We recall that the co-ordinate function $u_3(z)$ is harmonic in the annulus K_R and by virtue of Definition 1,

$$
(14) \qquad\qquad \lim_{z \to 1/R} u_3(z) = \tau_1, \qquad \lim_{z \to R} u_3(z) = \tau_2,
$$

where $\tau(\mathcal{M}) = (\tau_1; \tau_2)$ is the projection of the tube $\mathcal{M}$ onto x_3-axis.

We consider an auxiliary harmonic function

$$
h(z) = \tau_1 + \frac{\tau_2 - \tau_1}{2 \ln R} \ln |z|.
$$

It is easily seen that $h(z)$ satisfies (14). Thus $h_1(z) = u_3(z) - h(z)$ is harmonic in the annulus and

$$
\lim_{z \to \partial K_R} h_1(z) = 0.
$$

Then the maximum principle implies that $h_1(z) \equiv 0$ everywhere in K_R and hence

$$
(15) \qquad\qquad u_3(z) \equiv \tau_1 + \frac{\tau_2 - \tau_1}{2 \ln R} \ln |z|.
$$

In particular, it follows from (15) that

$$
du_3(z) \equiv \frac{\tau_2 - \tau_1}{\ln R} \cdot \frac{z}{|z|^2}
$$

doesn't vanish in K_R. We have, as a consequence, the normal $n(z)$ to $\mathcal{M}$ isn't parallel to e_3 at any point. Taking into account the above remark about the geometrical sense of $g(z)$ we obtain that $g(z) : K_R \to \mathbb{C} - \{0; \infty\}$.

By comparing of (15) and (12) we deduce that

$$
(16) \qquad\qquad 2g(z)f(z) = \frac{\tau_2 - \tau_1}{2 \ln R} \cdot \frac{dz}{z}.
$$

In order to eliminate $\ln R$ from the latter equality we substitute (16) into (12), and after using (10) we obtain

$$
(17) \qquad\qquad \ln R = \frac{\pi(\tau_2 - \tau_1)}{J_3}.
$$

On substituting of the found relationship into (16) we arrive at the conclusion of the lemma. $\square$

4. Proof of Theorem 1

Let us denote $w = (J_1 + iJ_2)/J_3$. Combining Lemma 2, (12) and (9) we obtain

$$\int_{C_1} \frac{1 - g^2(\zeta)}{2g(\zeta)} \frac{d\zeta}{\zeta} = 2\pi w_1 i,$$

$$\int_{C_1} \frac{1 + g^2(\zeta)}{2g(\zeta)} \frac{d\zeta}{\zeta} = 2\pi w_2.$$

Simplifying the last expressions and denoting $w = |w| \cdot e^{i\theta}$, $g_1(z) = -e^{-i\theta}g(z)$ give the following system

$$\frac{1}{2\pi} \int_{C_1} \frac{g_1(\zeta)d\zeta}{\zeta} = |w|,$$

$$\frac{1}{2\pi} \int_{C_1} \frac{d\zeta}{g_1(\zeta)\zeta} = -|w|.$$

Applying Theorem 2 we arrive at the inequality

$$\ln R \leq \frac{\pi^2}{|w| + \sqrt{1 + |w|^2}}$$

where $|w| \equiv |J_1 + iJ_2|/|J_3| = \tan\alpha(\mathcal{M})$. Using (17) we obtain the required estimate and the theorem is proved.

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